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ATTITUDE CONTROLS

FOR SUN-ORIENTED SPACE STATIONS

SUMMARY REPORT

to

The National Aeronautics and Space Administration

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ABSTRACT

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This report summarizes the results of a study on the effect of launch parameters and choice of control criteria on the reaction control propellant requirements of an axisymmetric rotating manned space station. The study deals with attitude control of the station toward the sun, as influenced by secular gravity gradient torques and the annual motion of the earth around the sun. Analytical relationships are derived and the results of computer studies are presented. It is concluded that when reasonable misalignment angles are permitted and control is in an on-off manner, significant differences in propellant requirement will exist between the four quarters of any one year. It is also shown that the most practical approach to reaction jet actuation is by on-off control which will at certain times maintain the sun-pointing axis near the maximum permissible misalignment angle in order to minimize the secular gravity gradient torques.

AUTHOR

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I. INTRODUCTION

This report summarizes the results of a study on attitude control of an earth orbiting, manned space station. The study was restricted to a rotating vehicle whose spin axis is nominally sun-oriented. It was further restricted to an axisymmetric configuration, i.e., one in which the transverse moments of inertia (around axes which are normal to the spin axis) are equal. The above conditions are pertinent to a large manned orbiting station in which spin is used to provide artificial gravity and the spin axis is oriented toward the sun so as to facilitate solar energy conversion by means of solar cells and the maintenance of thermal balance of the station.

This study represents an extension of the analytical parts of the work previously done by Exotech Incorporated under contract NASw-543 (Refs. 3 and 6). It was found by Exotech, as well as by others working on attitude control of a large manned station, that the secular gravity-gradient torque is a major source of disturbance on the station, leading to large propellant requirements for attitude control purposes. The present study was therefore aimed at the examination of control methods which will minimize the propellant requirement to compensate for gravity gradient torques. The general basis for seeking such control methods is derived from the fact that, for given station and orbit parameters, the magnitude of gravity gradient torques can be accurately predicted and since it is the dominant effect on station attitude, its effects can therefore be anticipated. More specifically, this study was intended to explore two aspects of the problem. The first, denoted as Task A, dealt with the effect of launch parameters on propellant consumption. It aimed at establishing whether a suitable choice of time of day or date of launch might lead to a minimization of propellant needed to overcome secular gravity-gradient torques over periods of from 1 to 3 years. The second task, denoted as Task B, was broader in scope in that it dealt with the choice of control criteria and associated instrumentation for reaction jet actuation. The objective in both tasks has been to minimize propellant requirements due to secular gravity-gradient torques, including the effect of the annual motion of the earth around the sun. As shown in reference (6), these two effects account for about 55 per cent of the disturbances on the station attitude relative to the sun.

The basic approach in this study relates to the fact that the manned orbiting stations contemplated for the near future are likely to tolerate reasonable misalignment relative to the sun. Specifically, it was assumed that sun orientation is needed primarily for solar energy conversion by means of solar cells and that the penalty for misalignment from the sun would therefore be mainly inefficient operation of the solar cell array. For example, at a misalignment of 20 degrees from the sun the loss in electrical power output would be about 6 per cent. Assuming that the station requires in the order of 20 kilowatts of electrical power, a power loss of even 10 per cent (corresponding to misalignment of about 25 degrees) would call for extra weight in solar array of only a few hundred pounds. However, it appears that the availability of 20 degrees misalignment can be used to reduce propellant weight for a one year mission by a thousand pounds or more. The degree to which propellant consumption would actually be reduced depends upon how effectively the permissible misalignment is utilized.

As described in the sections which follow, permitting a misalignment of the station's sun pointing axis offers the opportunity to reduce propellant requirements by making a suitable choice of the time of year of launch and by selecting an appropriate control policy for actuation of reaction jets.

II. GENERAL DISCUSSION

This section contains a discussion of analytical and numerical aspects of the study which are common to the two tasks.

A. Gravity-Gradient Torque

A rotating station with a high angular momentum along the spin axis is quite insensitive to the variations in gravity-gradient torque during one orbit. This is more fully described in reference (6) where it is shown that for the assumed station parameters the peak misalignment due to the periodic gravity-gradient torque would be in the order of 0.1 degree. This study therefore concerned itself only with the secular or average gravity-gradient torque per orbit.

Figure 1 defines the various coordinate systems and nomenclature used in the derivations of Appendix A. For the axisymmetric station, the secular gravity-gradient torque is given by equation (22) of Appendix A.

$$\vec{L}_s = \frac{1}{2} C_o (\vec{\xi} \cdot \vec{u})(\vec{\xi} \times \vec{u}) \tag{1}$$

$$C_o = 3 \omega_o^2 \Delta I \tag{2}$$

where ω_o = radian frequency for a circular orbit

ΔI = difference between the spin axis and transverse moments of inertia

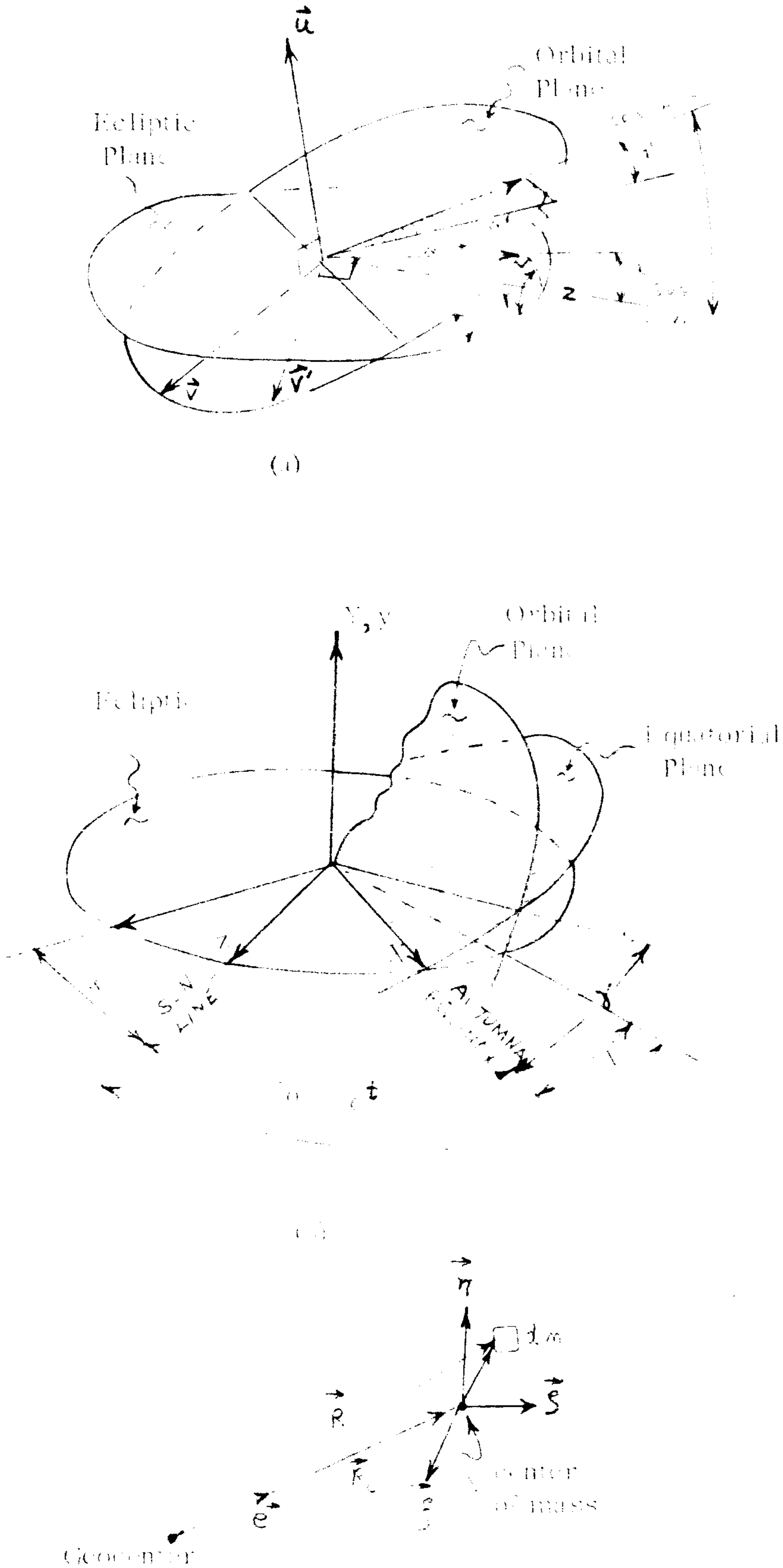
$\vec{L}_s$ = secular gravity-gradient torque

$\vec{\xi}$ = unit vector along station spin axis (axis of symmetry)

$\vec{u}$ = unit vector normal to the orbital plane (see Figure 1)

It is of particular interest to note that the secular torque $\vec{L}_s$ must always lie in the orbital plane since it is perpendicular to $\vec{u}$. Furthermore the magnitude of $\vec{L}_s$ may be defined as

$$|\vec{L}_s| = \frac{1}{4} C_o \sin 2\alpha \tag{3}$$



$\vec{u}, \vec{v}, \vec{w}$ - Geocentric ... orbital plane. ... is intersection of orbital plane with plane passing thru sun-line and ⊥ to orbital plane. $\vec{v}$ forms right-handed orthogonal set in order $\vec{u}, \vec{v}, \vec{w}$.

$\vec{u}, \vec{v}', \vec{w}'$ same as above except that w' relates to spin-axis $\vec{\xi}$ instead of sun-line.

X, Y, Z Inertial, with X and in ecliptic plane.

x, y, z Geocentric, with z in direction of the sun.

$\vec{\varsigma}, \vec{\eta}, \vec{\xi}$ - principal co-ordinates with origin at center of mass.

$\vec{e}$ - unit vector along $\vec{R}_c$

$\vec{R}_c$ - vector from geocenter to center of mass

$\vec{\xi}$ - directed along axis of symmetry with sense as defined by spin velocity vector.

FIG. 1 - CO-ORDINATE SYSTEMS AND NOMENCLATURE

where "a'" is the angle between $\vec{\xi}$ and w', viz. it is the angle between the spin axis and the orbital plane as measured in a plane normal to the orbital plane. These relationships are shown in Figure 2.

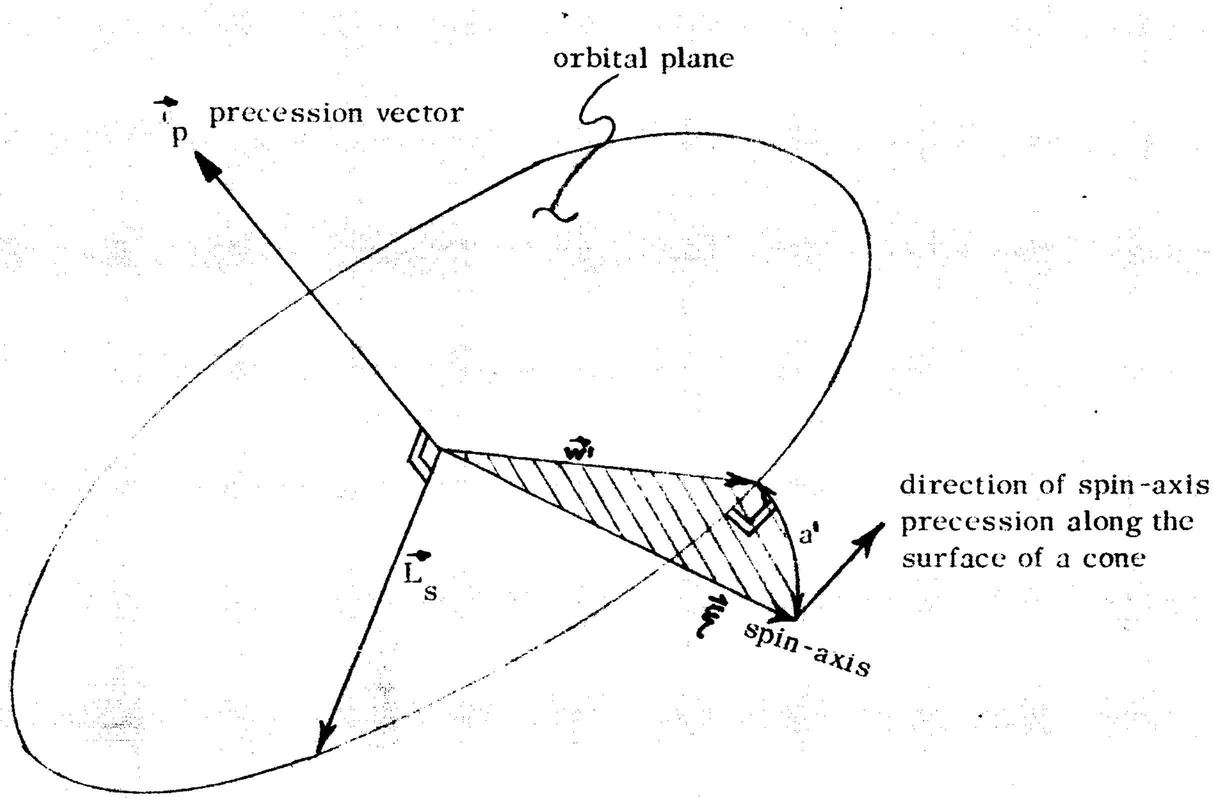
For a spinning station, it may be concluded that the secular gravity-gradient torque cannot, by itself, produce motion of the spin-axis which will change the magnitude of the torque. Since $\vec{L}_s$ lies in the orbital plane, the gyroscopic precession vector must be normal to the orbital plane and the motion of the spin axis will occur along the surface of a cone whose axis is normal to the orbital plane. The angle "a'" is therefore not altered and L_s would remain constant. If the orbital plane were to maintain a fixed inertial orientation, the spin axis would describe a cone around the orbital plane vector and would precess at a constant rate. The actual time variation of the secular gravity gradient torque is therefore caused only by regression of the orbital plane and the motion of the earth around the sun.

Figure 3 illustrates the annual changes of the secular gravity-gradient torque for the assumed orbit inclination of 28 degrees and the initial conditions indicated. This vector torque magnitude has been calculated on the assumption that the spin axis is continuously aligned to the sun. The control impulse needed to accomplish this alignment is obtained by evaluating the area under the torque curve without regard to sign.

Referring to Figure 3 it is to be noted that the torque curve can be considered to consist of a 54 day cyclic variation superimposed upon the dotted curve which is essentially sinusoidal and has a period of approximately one year.

B. Combined Effect of Annual Precession and Secular Gravity-Gradient Torque

If considered independently, the annual rotation of the earth around the sun would require the application of a control torque of constant magnitude so as to maintain sun alignment. As shown in reference (6), propellant requirements associated with the annual rotation would be second only to that due to the secular gravity gradient torque if the two are evaluated independently. It is, however, not realistic to consider these two effects independently because their very different characteristics suggest that the two would combine in a very particular way.



$$|\vec{L}_s| = \frac{1}{4} C_0 \sin 2a'$$

FIG. 2 - SECULAR GRAVITY-GRADIENT TORQUE GEOMETRY

Orbit Inclination: 28^{o}

Initial (Launch) Conditions: $\psi_o = \gamma_o = 180^{o}$

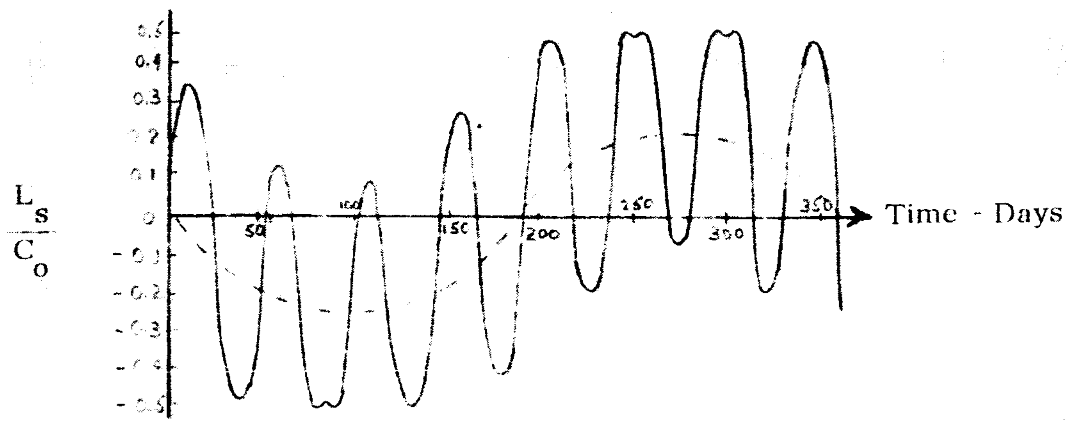


FIG. 3 - VARIATION OF SECULAR GRAVITY-GRADIENT TORQUE
FOR CONTINUOUS SUN ALIGNMENT

Orbit Inclination: 28^{o}

Initial (Launch) Conditions: $\psi_o = \gamma_o = 180^{o}$

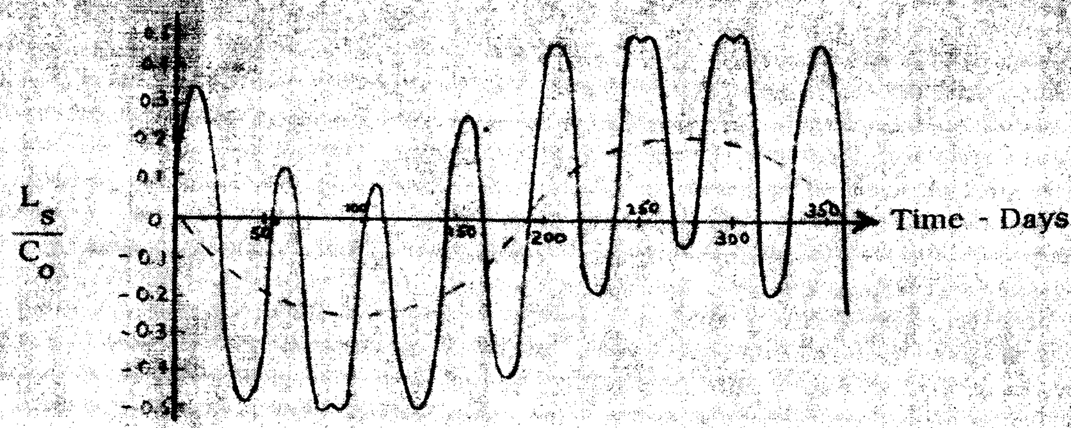


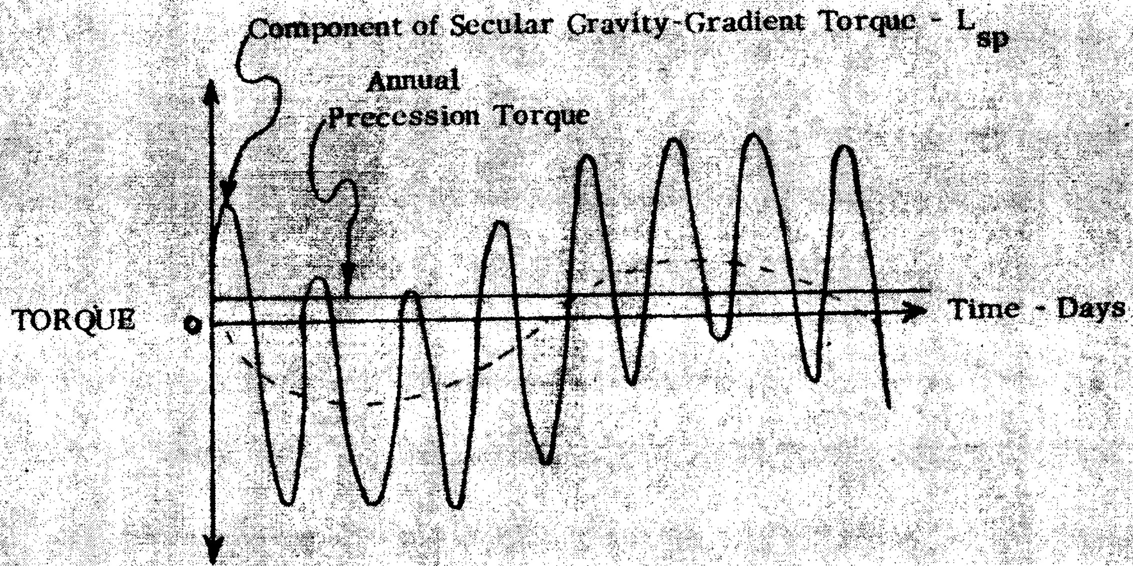
FIG. 3 - VARIATION OF SECULAR GRAVITY-GRADIENT TORQUE FOR CONTINUOUS SUN ALIGNMENT

To illustrate, we consider the case of continuous alignment of the spin axis to the sun. The total secular torque acting on the station has been shown in Figure 3 and the component of this secular torque L_{sp} which produces precession of the spin axis about the same axis as the annual rotation of the orbit around the sun is shown in Figure 4. Also shown in Figure 4 is the constant and unidirectional annual precession torque applicable to the assumed station parameters as defined in Section II-D below. To maintain sun alignment, the torque required at any instant of time would be the algebraic sum of the above two torques.

It can be seen from Figure 4 that the time variation of the secular gravity-gradient torque component L_{sp} can be described by a sinusoid with a period of about 54 days superimposed upon a sinusoid with a period of about one year. We consider first the way in which the two torques combine in a single 54 day cycle. As illustrated in Figure 5, if the sinusoidal curve were symmetrically located about the time axis, it is evident that the difference between the area under the absolute value of L_{sp} and the area taken under $|L_{sp} + L_p|$ would be given by the small areas A_1 and A_2. Thus, except for area $A_1 + A_2$, the accumulated impulse due to the combined effect of the secular gravity-gradient torque and the annual precession could be found neglecting the latter.

Since the 54 day cycles are generally not symmetric about the time axis each cycle will produce different values of the differential area $A_1 + A_2$. However, because there is symmetry in these variations between the first and second half of the year, these differences will tend to cancel out over a one year interval.

The above leads to two conclusions regarding the combined effect of annual precession and secular gravity-gradient torques. First, if propellant requirements are to be computed for a period of one year or multiples of one year, the total requirement can be estimated by considering the secular gravity-gradient torque alone. Secondly, inclusion of the annual motion of the earth can be expected to influence the rate of propellant consumption during the year. Specifically, its effect would be to increase propellant requirements during one half of the year and to decrease it during the other.



Initial Conditions: $\psi_o = \gamma_o =$ [illegible]

$i = 28^o$

Note: This plot was taken in part from reference (5).

FIG. 4 COMPONENT OF GRAVITY-GRADIENT TORQUE

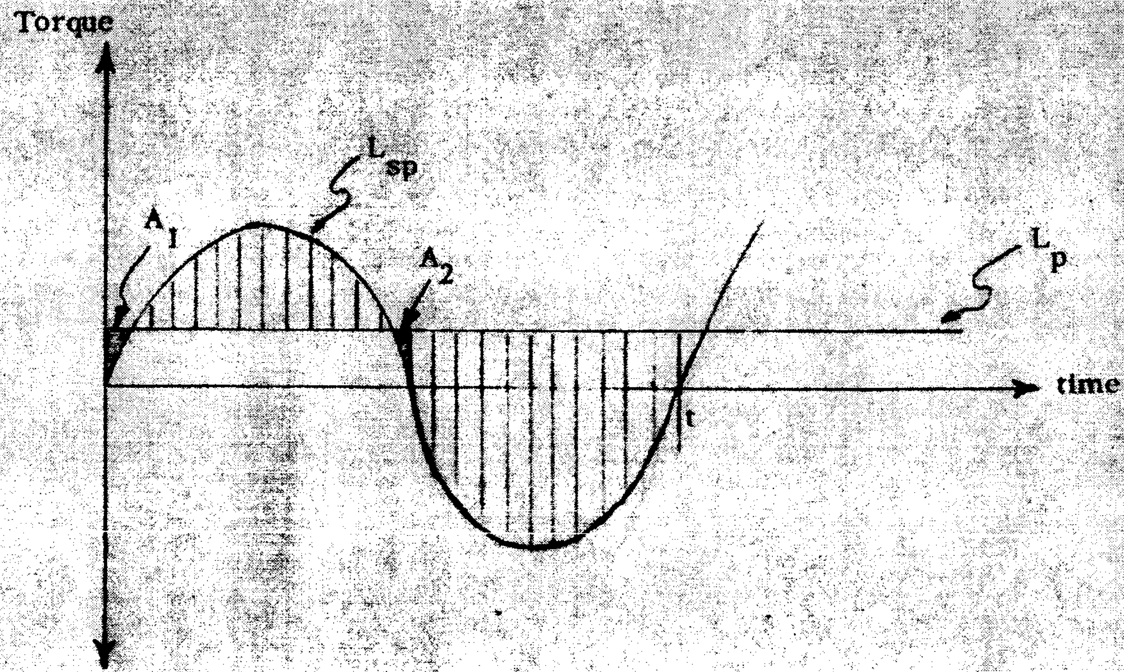


FIG. 5 - ILLUSTRATION OF COMBINED EFFECT OF GRAVITY-GRADIENT AND ANNUAL PRECESSION

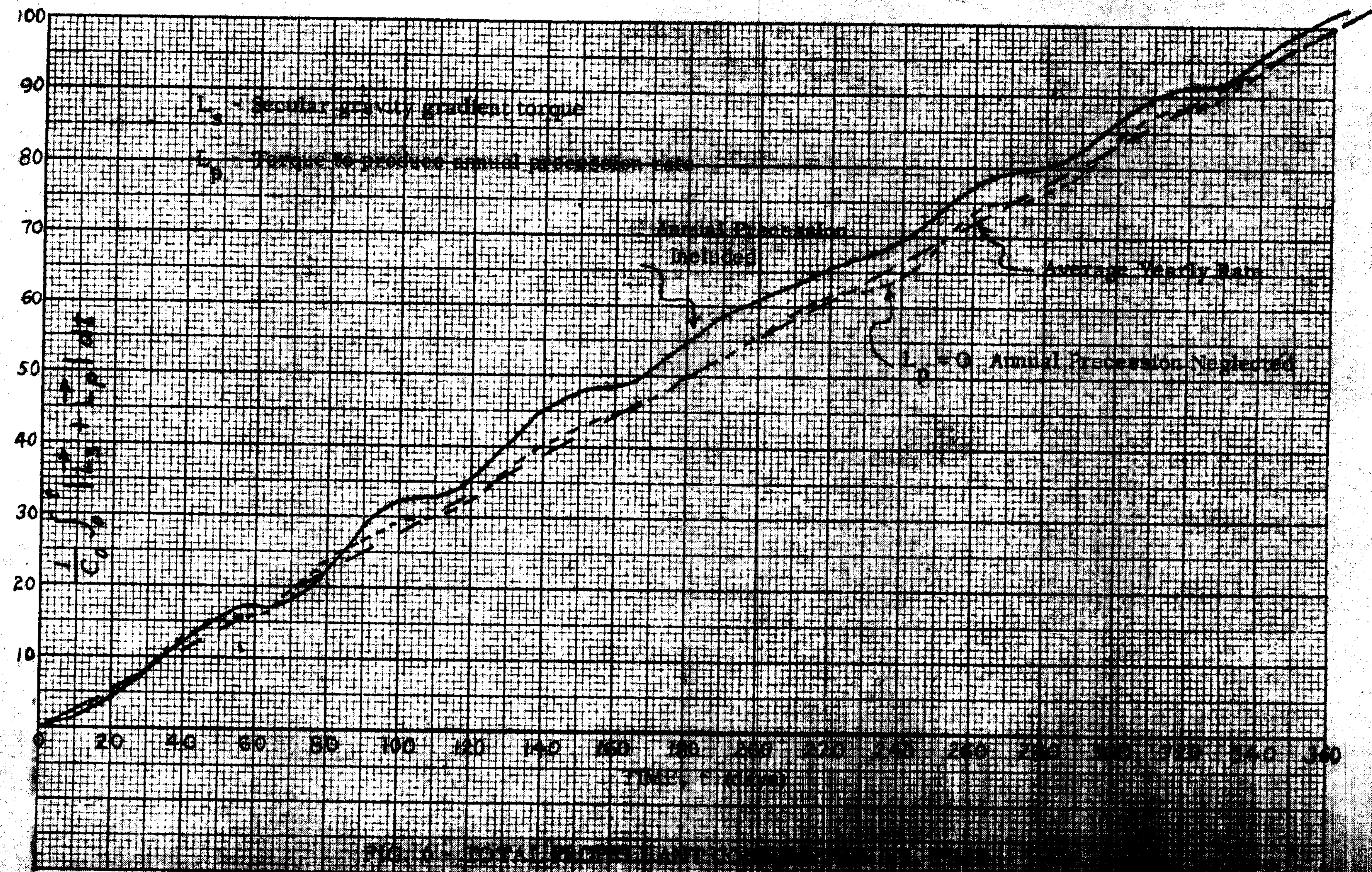
The validity of the above conclusions is substantiated by Figure 6 in which the cumulative integral of torque is plotted as a function of time, i.e., the ordinate on the curve is proportional to the propellant which would be required up to the time considered. Thus, a straight line drawn from the origin to t = 365 days will give the average rate of propellant consumption during the year. Since the dotted curve based on gravity-gradient torque alone stays close to this straight line, the corresponding average rate of propellant consumption would be about the same throughout the year. Referring now to the solid curve, representing the case where annual precession and secular gravity-gradient torques are considered simultaneously, it can be seen that (1) the value at the end of one year is about the same as for the case where annual precession was neglected (the difference is about 2.5%) and (2) the rate of propellant consumption is somewhat larger than the average during the first half of the year and smaller during the second half.

C. Attitude Dynamics

The assumption that the spin axis is continuously aligned to the sun simplifies both analysis and numerical calculation and is therefore useful in considering general characteristics such as those discussed above. However, in the actual situation, the spin axis will be misaligned, the extent and nature of the misalignment being dependent upon the control policy which is used. To permit a more realistic evaluation in both of the study tasks, the dynamics of spin axis motion relative to the sun must therefore be included.

Equations (31) thru (40) of Appendix A are the equations used in evaluating propellant requirements when the spin axis is controlled to the sun in an on-off manner. The following basic assumptions are used in the derivation and application of these equations.

(a) The spin angular momentum is large compared with the maximum gravity-gradient torque-impulse during one quarter of an orbit so that the average gravity-gradient torque during an orbit, rather than the instantaneous value, can be used in the calculation of spin axis drift.



L_s - Secular gravity gradient torque
Annual Precession Included
Average Yearly Rate
$L_p = 0$ Annual Precession Neglected
$\frac{1}{C_o}\int_o^t \left| \frac{L_s}{4} + \frac{L_p}{4} \right| dt$
0
10
20
30
40
50
60
70
80
90
100
20
40
60
80
100
120
140
160
340
360

(b) The station possesses axial symmetry about the spin axis.

(c) Angular momentum and angular velocities due to the gravity-gradient torque about axes normal to the spin axis are negligibly small compared to the spin angular momentum. (Compare maximum drift of 10 degrees/day with 3 revolutions per minute spin.) This permits the use of the simple dynamic equation:

$$\vec{L}_s = I_s \nu_s \frac{d\vec{\xi}}{dt} \tag{4}$$

$\vec{L}_s$ = secular gravity-gradient torque

I_s = spin axis moment of inertia

ν_s = spin angular velocity

$\frac{d\vec{\xi}}{dt}$ = precession rate

(d) It is assumed that corrective torque impulses are delivered instantaneously and that damping will be provided to damp out any transients in a time short relative to the time of free drift. Hence, the dynamic equation represents only the steady precessional motion of the spin axis.

Application of the dynamic equations also required the use of the various trigonometric relationships which relate the orbital parameters as a function of time of year. These will be found in reference (6).

Two types of on-off control policies for the application of corrections were considered in the computer solution, namely (a) drift of the spin axis to some predetermined sun deviation angle δ and then correction of the spin axis back to the sun line (full correction) and (b) drift to some predetermined angle δ and then correction of the spin axis back to some predetermined angular deviation δ_1, (partial correction).

In both cases, the computer solution is really a series of solutions of the dynamic equations with initial conditions determined by specifying the direction of the spin vector in inertial coordinates just after each correction. Each correction implies the application of a control torque impulse by use of propellant in a time small compared with the drift times.

Since a series of initial value problems are solved, it proved convenient to select a new inertial coordinate system (corresponding to the instantaneous sun line) at the beginning of each drift interval. For full corrections, the spin axis is along the instantaneous sun line at the beginning of each drift interval. For partial corrections the spin axis was moved along the shortest path between the outer limit cone (δ) and the inner limit cone (δ_1) i.e., along a "radial" path toward the sun.

Figures 7 and 8 summarize results of computations using a Remington Rand Solid State 80 digital computer and based upon the assumed parameters as defined in Section II-D below. Each point in Figures 7 and 8 represents the time and angular deviation at which a control torque impulse was applied. Since integration was carried out using average parameter values for each one day interval, the desired angle at which the correction was applied could only be approximately controlled. For example, for on-off control to 10 degrees, the computer was programmed to reset to zero angular misalignment whenever the misalignment exceeded 9 degrees. Thus, depending upon the magnitude of the secular torque during the one day interval and its direction relative to the effect of annual precession, the actual angle at which the "control correction" was applied varied as shown in Figures 7 and 8.

D. Assumed Station Parameters

The following numerical values of orbit and station parameters were used throughout the study:

(a) Spin axis moment of inertia - $I_s = 1.5 \times 10^7$ slug-ft.^{2}

(b) Transverse moments of inertia (equal) - 1.05×10^7 slug-ft.^{2}

$$\Delta I = 4.5 \times 10^6 \text{ slug-ft.}^2$$

(c) Moment-arm of reaction control nozzle - 75 feet

(d) Propellant specific impulse - I_{sp} = 300 sec.

(e) Spin rate - 3 RPM; angular velocity vector pointing to the sun

(f) Orbit altitude - circular at 300 nautical miles

(g) Orbit inclination to equator - 28 degrees

(h) Initial orbital parameters - $\gamma_o = 180^o$; $\nu_o = 180^o$

Assumptions (c) and (d) were needed only for estimating propellant weight in pounds. Weights thus estimated do not include the weight of propellant tankage.

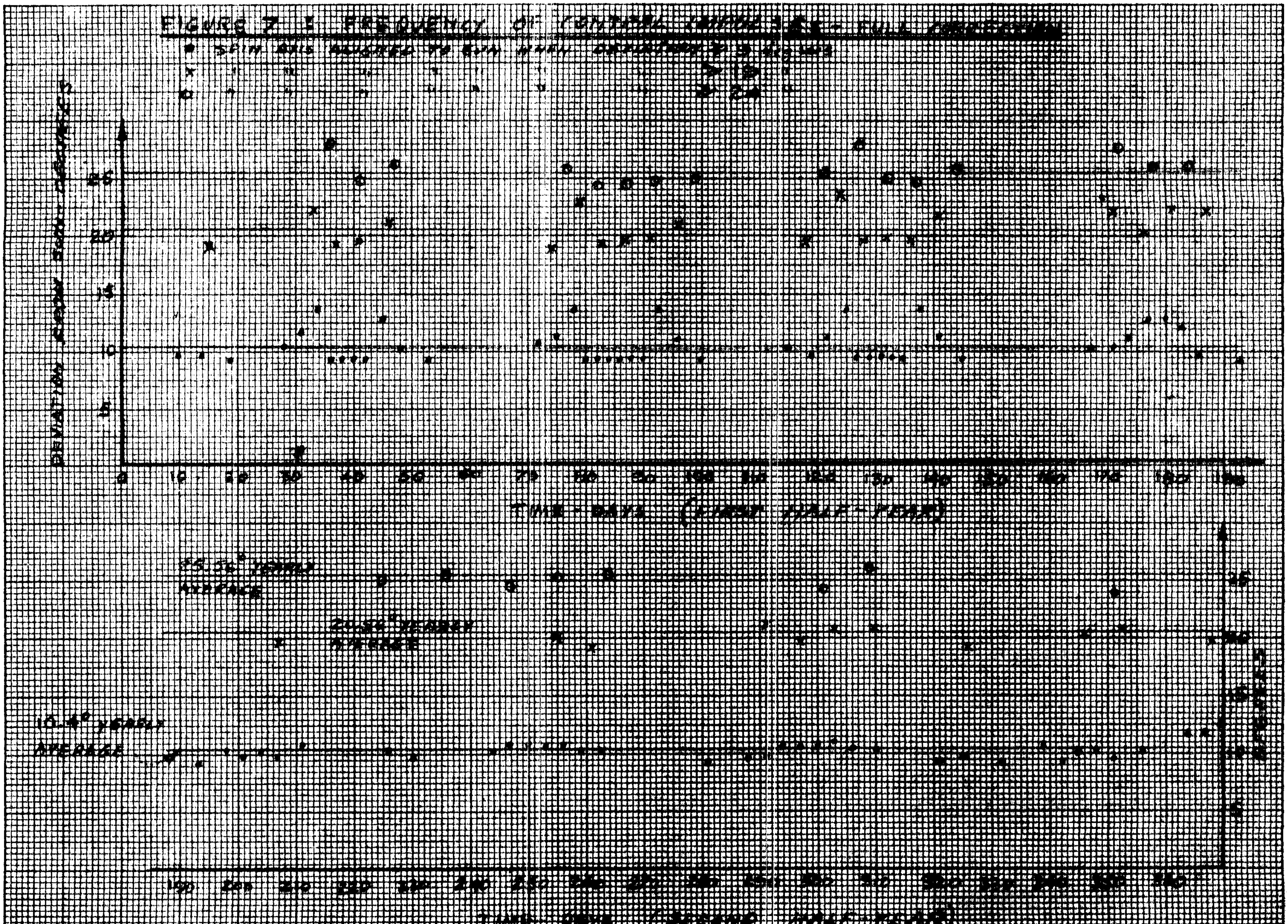
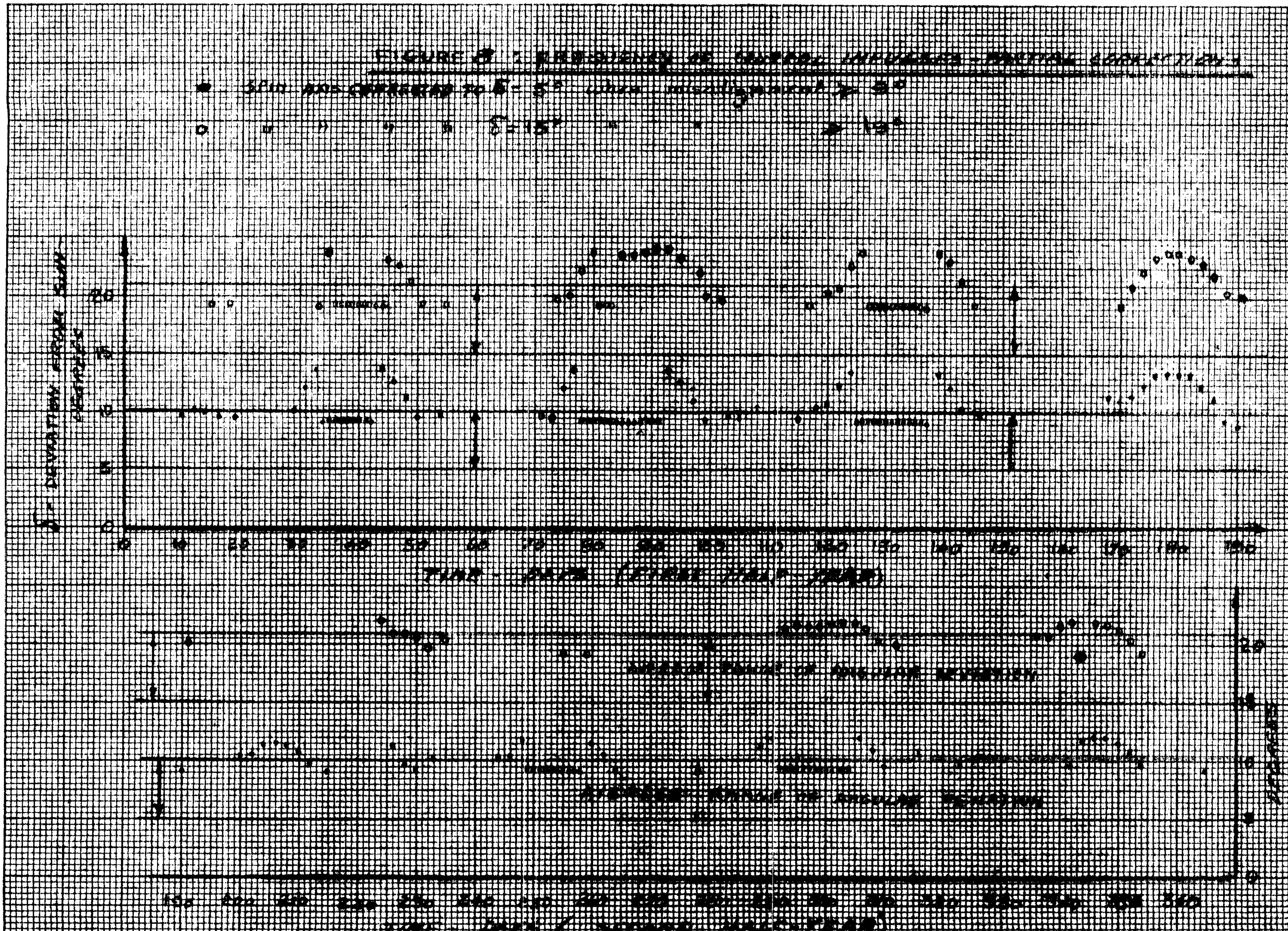


FIGURE 7.1 FREQUENCY OF CONTROL
DEVIATION FROM SUN - DEGREES
TIME - DAYS (FIRST HALF-YEAR)
25.56° YEARLY AVERAGE
21.56° YEARLY AVERAGE
10.4° YEARLY AVERAGE
TIME - DAYS (SECOND HALF-YEAR)


δ - DEVIATION FROM SUN
DEGREES
0
5
10
15
20
TIME - DAYS (FIRST HALF-YEAR)
AVERAGE RANGE OF ANGULAR DEVIATION
DEGREES
TIME - DAYS (SECOND HALF-YEAR)

III TASK A

EFFECT OF LAUNCH PARAMETERS ON PROPELLANT CONSUMPTION

Referring to Fig. 1, the analytical parameters which correspond to a given date and time of day at which the space station is injected into orbit are the angles ψ_o and γ_o. It is assumed that the problem starts after the spin-axis has been aligned to the sun so that ψ_o defines the angle between the spin-axis and the autumnal equinox as measured in the ecliptic plane and γ_o is the angle between the orbital line of nodes and the autumnal equinox as measured in the orbital plane. In order to determine the effect of the initial launch parameters on propellant consumption, it is necessary to evaluate the cumulative integral of the applied corrective torques. To do this, it is also necessary to define how control torques are applied to the station so that the path which the spin-axis will describe and the corresponding magnitudes of secular gravity-gradient torques can be calculated.

The simplest approach to this problem would assume that control torques are applied to the station in the proper direction and magnitude, so as to counteract continuously the secular gravity-gradient torque and maintain alignment with the sun. As a further simplification, the presence of annual earth motion could also be neglected. Under these limiting conditions, the problem reduces simply to the integration of the secular torque over a prescribed interval of time for different initial conditions, i.e., for different values of ψ_o and γ_o. Restrictive as these assumptions may be, the results are of interest in that they provide a reference for the more realistic situations.

The above approach has been taken in the study of Reference (4) and a review of the results, presented in References (4) and (5), showed that, for a one year interval, propellant consumption did not vary by more than a few percent for seven different combinations of initial launch conditions. These results are consistent with a qualitative evaluation of the variation of the secular gravity-gradient torque with time. As shown by the dotted line in Fig. 3, the torque curve tends to repeat itself at intervals of about

365 days. This annual variation is produced by the earth's motion around the sun. It thus appears that the choice of 365 days as the basis for evaluating long term variations of the torque integral tends to obscure the problem since the effect of different initial conditions tends to be averaged out over a one year period.

In view of this observation, it is of interest to establish how propellant consumption might vary during the course of a one year period. The results of a calculation for the assumed conditions of continuous sun alignment and omission of annual precession are contained in Fig. 6 and it is seen that the total propellant consumption for periods of a few weeks or more is essentially the same as the average yearly rate. In conclusion, therefore, the choice of launch parameters has no significant effect either on short term (a few weeks) or long term propellant consumption if one assumes continuous sun alignment and particularly when the effects of annual precession are omitted.

It has been shown in Section II-B that when the gravity-gradient torque is considered simultaneously with annual precession the principal effect of the latter is to unbalance propellant consumption during the year without significantly influencing the total amount consumed during one year. Emphasis in this study was therefore placed on determining in more detail the extent to which propellant rates will vary during the course of one year when the station spin-axis is controlled in an on-off fashion to varying degrees of angular accuracy.

The computer data summarized in Figures 7 and 8 are directly applicable to this problem since the angle through which the spin-axis is reoriented at each correction is directly proportional to the propellant required to perform the maneuver. Results of calculations based on these computer data are summarized in Table I.

Referring to Table I, it is seen that the unbalance in propellant consumption between the two halves of one year becomes more pronounced as the accuracy of spin-axis alignment relative to the sun is reduced. This effect is attributed to the annual precession required to maintain sun-orientation. As will be shown in the discussion of Task B, reduced sun-pointing accuracy leads to reduced yearly propellant consumption which implies lower average levels of secular gravity-gradient torques. Referring to Fig. 5, it can be seen that as the peak value of the cyclic curve (representing the gravity-gradient torque) becomes smaller in relation to the straight horizontal line (representing the effect of annual precession) the ratio of the areas A_1 and A_2 to the area of $|L_s + L_{sp}|$ increases. Thus, the effect of the annual precession becomes relatively more dominant and the unbalance in propellant rates during the two halves of the year is correspondingly increased.

Table I also indicates a quarterly variation in propellant rates which is, however, not consistent on the basis of the consecutive quarters used to present the data. These variations may be explained qualitatively as follows:

Referring to Fig. 2 and equation (3), it is evident that for orbit inclination angles of the order of 28^o and assuming the spin axis to be in the ecliptic plane (i.e. for continuous alignment to the sun), the peak values of secular torque developed during a regression cycle would depend upon the time of year. Thus, one might expect the peaks to be minimized during the period near the Vernal Equinox (March 21) and Autumnal Equinox (September 21) when the inclination of the sun-pointing spin-axis to the orbit plane can be no greater than the orbital inclination angle. On the other hand, at the winter and summer Solstices, the inclination of the equatorial plane to the sun line is a maximum and the orbital inclination angle can add directly to the angle between the equatorial and ecliptic planes. Therefore, at these times, the angle (a) between the spin axis and the orbital plane can reach its maximum value and the secular torque would consequently reach its maximum peak value. In general, therefore, one might expect propellant consumption to be lower at times near the Vernal and Autumnal Equinoxes and higher during the winter and summer Solstices. This is borne out by the shape of the secular torque curve in Fig. 3 where $t = 0$ and $t = 182.5$ days correspond to the Equinoxes and $t = 91.2$ and $t = 273.8$ days correspond to the two Solstices. This argument also tends to explain why there would be differences between the quarterly propellant rates in Table I.

To illustrate the significance of the above variations, we assume a mission duration of 18 months and consider two choices of launch conditions. We also assume that in both instances on-off control is used to realign the spin-axis to the sun when the misalignment angle has reached a value of 20^o. If launch conditions were the same as those used in the computer runs, the total propellant requirement for 18 months would be 165 percent of the yearly requirement. If, on the other hand, initial conditions were such as to correspond

TABLE I

VARIATION OF PROPELLANT REQUIREMENTS DURING THE YEAR FOR VARIOUS CONTROL POLICIES

Period of the Year	0^o	$10^o \rightarrow 0^o$	$10^o \rightarrow 5^o$	$20^o \rightarrow 0^o$	$20^o \rightarrow 15^o$	$25^o \rightarrow 0^o$
	Percent of Yearly Requirement For Range of Angular Misalignment					
First Half	53.5%	56%	56.7%	64.9%	64.1%	65.5%
Second Half	46.5	44	43.3	35.1	35.9	34.5
First Quarter	25.0	28.2	28.0	31.4	29.1	26.0
Second Quarter	28.5	27.8	28.7	33.5	35.0	39.5
Third Quarter	24.8	20.3	23.1	12.6	12.6	22.5
Fourth Quarter	21.7	23.7	20.2	22.5	23.3	12.0
Largest Quarter		31.1		39.6		41.4
Smallest Quarter		19.6		9.3		12.0

to t = 180 days in the computer runs, the total requirement would be 135 percent of the yearly requirement. This gives a difference of 30 percent of the yearly requirement. For the assumed station parameters, this is equivalent to a difference of 680 lbs. For a short duration mission of the order of three months Table I shows that the ratio between the most favorable and least favorable quarter is more than 4:1. Stated differently, the wrong choice of initial conditions can lead to an excess propellant requirement of as much as 680 lbs. for a three month mission. (The fact that the variability of possible propellant requirements in this example has the same range, 680 lbs., for both an 18-month and a three-month mission is coincidental, and is associated with the assumed parameters.)

IV TASK B

INSTRUMENTATION FOR CONTROL OF REACTION JET ACTUATION

The principal objective of the studies conducted under this task has been to evolve an approach to control instrumentation which will best utilize the permissible angular misalignment from the sun. "Best" is intended to mean an optimum choice between on-board instrumentation complexity and propellant consumption. Thus, the best approach to reaction-jet control would achieve a substantial propellant reduction with little increase in instrumentation complexity.

A suitable reference for evaluating different control methods is the propellant needed to orient the spin-axis continuously toward the sun, i.e., without any angular misalignment. This reference propellant requirement is obtained simply by integrating under the secular torque curve and the value calculated for a one year interval is taken as the 100 percent reference propellant requirement. For the typical station parameters, the corresponding weight of propellant, including the small differential to produce the annual precession, is 3,180 lbs.

Three different control policies, with increasing instrumentation complexity, were evaluated. They include: (1) simple on-off control with the control torque returning the spin-axis to exact alignment with the sun, (2) on-off control with partial corrections, i.e., where the control torque moves the spin-axis toward the sun but only part of the way, and (3) continuous control, applied so as to maintain the spin-axis along a predetermined path within the cone described by the permissible angular misalignment from the sun. A more detailed description of these three control policies and the results of their evaluation follows:

A. On-Off Control With Full Correction

This approach involves minimum instrumentation complexity. Misalignment of the spin-axis from the sun is measured by means of sun sensors and a control torque is applied to reduce the sun error angle to zero whenever the misalignment reaches a preselected maximum value. Between corrections the spin-axis is allowed to drift freely. The calculation of propellant requirements assumes that only a single control impulse is applied when the control nozzle (or pair of nozzles) is 90 degrees displaced from the projection of the sun line in the plane of the station. The torque would then precess the spin-axis toward the sun. In this case, it is assumed that the magnitude of the control impulse will be such as to return the spin-axis to exact alignment with the sun. (Actually a sequence of control impulses may be used with one impulse during each rotation of the station over a period of minutes or hours.) A detailed discussion of control equipment and logic circuitry for this control method may be found in Reference (4).

The computer data summarized in Fig. 7 can be used to calculate yearly propellant requirements for this control policy. Using the yearly requirement for continuous alignment as the reference value, the reduction due to increased angular misalignment was found to be as follows:

Permissible Angular Deviation Degrees	Yearly Propellant Requirement
0	100%
10^{o}	91%
20^{o}	71.5%
25^{o}	69%

There are two possible reasons for the reduced propellant requirement shown above. First, as the spin-axis drifts from the sun the angle (a') (Fig. 2) will be different than if the spin-axis were always in the ecliptic plane, as is the case for $\delta = 0$. Although in the course of the year the change in (a') may be sometimes favorable and at other times unfavorable, the net effect probably tends to reduce the average value of the secular gravity-gradient torque acting on the station. Since a numerical comparison of the secular torques in the two cases was not made, the above conclusion is speculative. The second reason, however, is more definitive and is explained as follows:

When the control jets are used "continuously" (i.e. on every rotation of the station) to apply an average torque to the station which is equal and opposite to the disturbing torques, the corresponding impulse supplied by the propellant is given by

$$I_c = \int_{t_2}^{t_2} |\vec{L}_d| \, dt \tag{5}$$

I_c - control impulse

$\vec{L}_d$ - total disturbing torque

Under drift conditions the appropriate equation is:

$$\frac{d\vec{H}}{dt} = \vec{L}_d = I_s \omega_s \frac{d\vec{\xi}}{dt} \tag{6}$$

and

$$\Delta \vec{H} = \int_{t_1}^{t_2} \vec{L}_d \, dt = I_s \omega_s \Delta \vec{\xi} \tag{7}$$

In order to return the spin-axis to the sun line, a control torque L_c must be supplied to make $\Delta \vec{\xi} = 0$, requiring an impulse

$$I_\xi = |I_s \omega_s \Delta \vec{\xi}| = \left| \int \vec{L}_c \, dt \right| \tag{8}$$

$$I_\xi = \left| \int_{t_1}^{t_2} \vec{L}_d \, dt \right| \tag{9}$$

Since in general

$$\left| \int_t^{t+\Delta t} \vec{A} \, dt \right| \leq \int_t^{t+\Delta t} |\vec{A}| \, dt \quad \text{where A is any vector} \tag{10}$$

it follows that

$$I_\xi \leq I_c \tag{11}$$

Thus, the total control torque impulse under drift conditions can generally be expected to be smaller than that required to maintain continuous alignment to the sun.

B. On-Off Control With Partial Corrections

The two causes of reduced yearly propellant requirement in the preceding case are derived from the fact that the spin-axis is permitted to drift within the limit of permissible angular deviation from the sun. It is, however, not known whether the particular control mode just considered leads to optimum drift conditions. In fact, one might conjecture that since propellant consumption decreases with the magnitude of the deviation angle, keeping the spin-axis near the maximum allowable misalignment might lead to further reduction in propellant requirement. To test this possibility, two computer runs were made with only partial corrections. Thus, the spin-axis was permitted to drift to the maximum permissible angles of 20° in one run and 10° in the other, and when this value was reached the computer was programmed to reset to 15° and 5° respectively rather than to zero as in the preceding case. Each correction was such as to move the spin axis radially towards the sun. The data from these two runs are summarized in Fig. 8. (As previously noted, because of the discrete integration interval the actual program called for a reset when $\delta > 19^{\circ}$ and $\delta > 9^{\circ}$ respectively leading to average values of the misalignment angle of 20.8° and 10.2° at the time of correction.)

The results of the two runs with partial corrections are shown below. For comparison, propellant requirements for the corresponding cases of full correction are also included:

Range of Angular Deviation-Degrees	Yearly Propellant Requirement
0	100%
0-10	91%
5-10	78.8%
0-20	71.5%
15-20	61.0%

Implementation of this control policy of partial corrections involves no added complexity over that for the full-correction method; the only difference is in the magnitude of the corrective impulses. Since the chosen range of deviation angles in the above two runs was quite arbitrary, no quantitative conclusion can be drawn from this data. However, further insight into the maximum possible reduction which this approach can yield is gained by comparing it to the third control mode, discussed below.

C. Continuous Control Along A Predetermined Path

If one were to disregard instrumentation requirements as a basis for judging different control policies, the control problem could be formulated mathematically as that of defining the path which the spin-axis should take within the limits of the permissible angular deviation from the sun so as to yield the minimum propellant consumption. To define the path analytically is not possible because a closed form solution of the equations involved is not available. Restricted forms of this "optimum path" solution can, however, be profitably considered.

Referring to equation 3 and Fig. 1, it can be shown that for a given permissible deviation from the sun δ, the secular gravity-gradient torque will be a minimum if δ is used to reduce the angle "a" by the maximum possible amount. This implies that the correction should be made by moving the spin-axis in the plane normal to the orbital plane. A mathematical proof is provided in Appendix B where it is shown that:

$$L_{s_{min.}} = \frac{C_o}{2} \sin 2(a \pm \delta) \tag{12}$$

The choice of the plus or minus sign depends upon the sign associated with the location of the spin-axis and the orbital plane relative to the ecliptic plane.

We may now assume an idealized situation in which the secular gravity-gradient torque just happens to move the spin-axis along the minimum torque path. Thus, when $a > \delta$ the spin axis is assumed to be displaced from the sun by the angle δ but always in a plane normal to the orbital plane, and when $a < \delta$ the spin axis is assumed to move in such a way as to stay in the orbital plane. During the latter part the secular torque will therefore be zero.

As described in Section II-A, the effect of the secular torque is to move the tip of the spin axis parallel to the instantaneous position of the orbital plane. Thus, if we assume that initially the desired minimum torque orientation exists, the effect of the secular torque would indeed be to rotate the spin-axis about the sun line holding the sun error angle constant. However, it can hardly be expected that either the direction of rotation and/or the amount of rotation will be such as to produce the assumed minimum torque conditions at all times. The idealized case is therefore not attainable in reality and its usefulness lies only in the fact that propellant requirements associated with it represent a lower limit for the more realistic situations. For having assumed that the desired minimum torque path will be traversed, the only function of the control system will be to overcome the secular torque associated with this path.

Fig. 9 illustrates the variation of the secular gravity gradient torque over a typical period of time under the conditions assumed above. As previously defined, all curves for which $\delta \neq 0$ have periods of zero torque when $a < \delta$. Propellant requirements are therefore found by taking the areas under each of these curves. Using $\delta = 0$, i.e., continuous sun alignment, as a reference, relative propellant requirements were calculated for the 52.5 day interval of Fig. 9. Since all other calculations are done on a one year basis, the above idealized conditions were also extended to one year. Results of these calculations are summarized below:

δ	Relative Propellant Requirement For 52.5 Day Period	Relative Propellant Requirement For One Year Period
0°	100%	100%
10°	62%	61%
20°	30%	34%

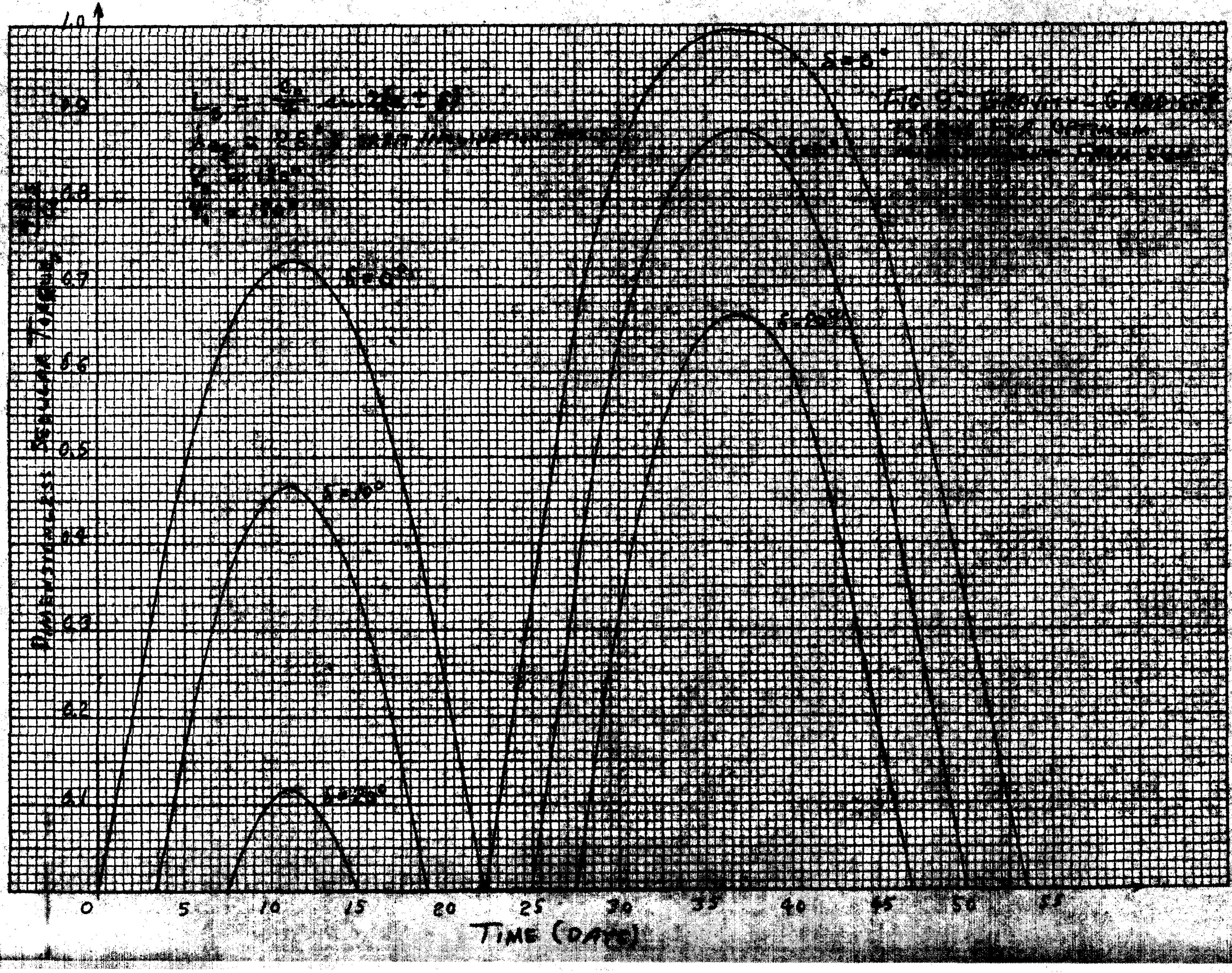
Since the preceding calculation defines the minimum possible propellant requirement for continuous control along an optimum path, it is of interest to evaluate next an upper limit for propellant requirement based on this control policy. This can be done by assuming, as before, that the desired path is that which produces at all times a minimum secular gravity-gradient torque. However, the propellant requirement will now be calculated to include the impulse needed to orient the spin-axis along this specified path. Such a calculation was performed for the 52.5-day period of Fig. 9 and for $\delta = 10^\circ$. The basic procedure was to use the equation of motion:

$$\vec{L} = I_s \nu_s \frac{d\vec{\xi}}{dt} \tag{13}$$

and

$$\vec{\xi} = \vec{I}\,\xi_x + \vec{J}\,\xi_y + \vec{K}\,\xi_z \tag{14}$$

where $\vec{\xi}$ in inertial coordinates may be specified from the geometry. A numerical (graphical) determination of $d\vec{\xi}/dt$ was used to evaluate the torque $\vec{L}$ which produces motion along the desired path. This torque consists, however, of the control torque


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
20
25
30
35
40
45
50
55
TIME (DAYS)

$\vec{L}_c$ and the secular gravity-gradient torque $\vec{L}_s$ so that

$$\frac{\vec{L}_c}{I_s \omega_s} = \frac{d\vec{\xi}}{dt} - \frac{\vec{L}_s}{I_s \omega_s} \qquad (11)$$

Since $\vec{L}_s$ may also be determined from the geometry knowing the path of the spin vector, $\vec{\xi}$, it is thus possible to find $\vec{L}_c$. The latter is a control torque which must be supplied continuously by the reaction jets; hence, the propellant requirement is obtained by plotting $\frac{L_c}{I_s \omega_s}$ versus time and evaluating the area under this curve.

The result of the above calculation for $\delta = 10^0$ showed that the saving, compared to continuous sun alignment, is about 1.3%. It thus appears that, at least for the case studied, propellant requirements to orient the spin axis along the path of minimum gravity-gradient would be about equal to the reduction in impulse due to the lower gravity-gradient torques. There is thus no net saving.

The above does not mean that control along some other preselected path will not result in larger savings. The actual optimum path would be one along which the gravity-gradient torque may be larger than the minimum possible, but the propellant required to maintain this path is reduced so that the combination of these two propellant requirements is an optimum one. The calculations described above merely indicate the range within which this approach to reaction jet control will fall.

In its general form, this approach calls for the greatest instrumentation complexity. Thus, it is first of all necessary to sense the inertial orientation of the spin-axis and to compare this with a program of the desired inertial orientation. In addition, the application of corrective torques must also be done in terms of inertial coordinates, which requires an inertial sensing system not needed for control policies A or B.

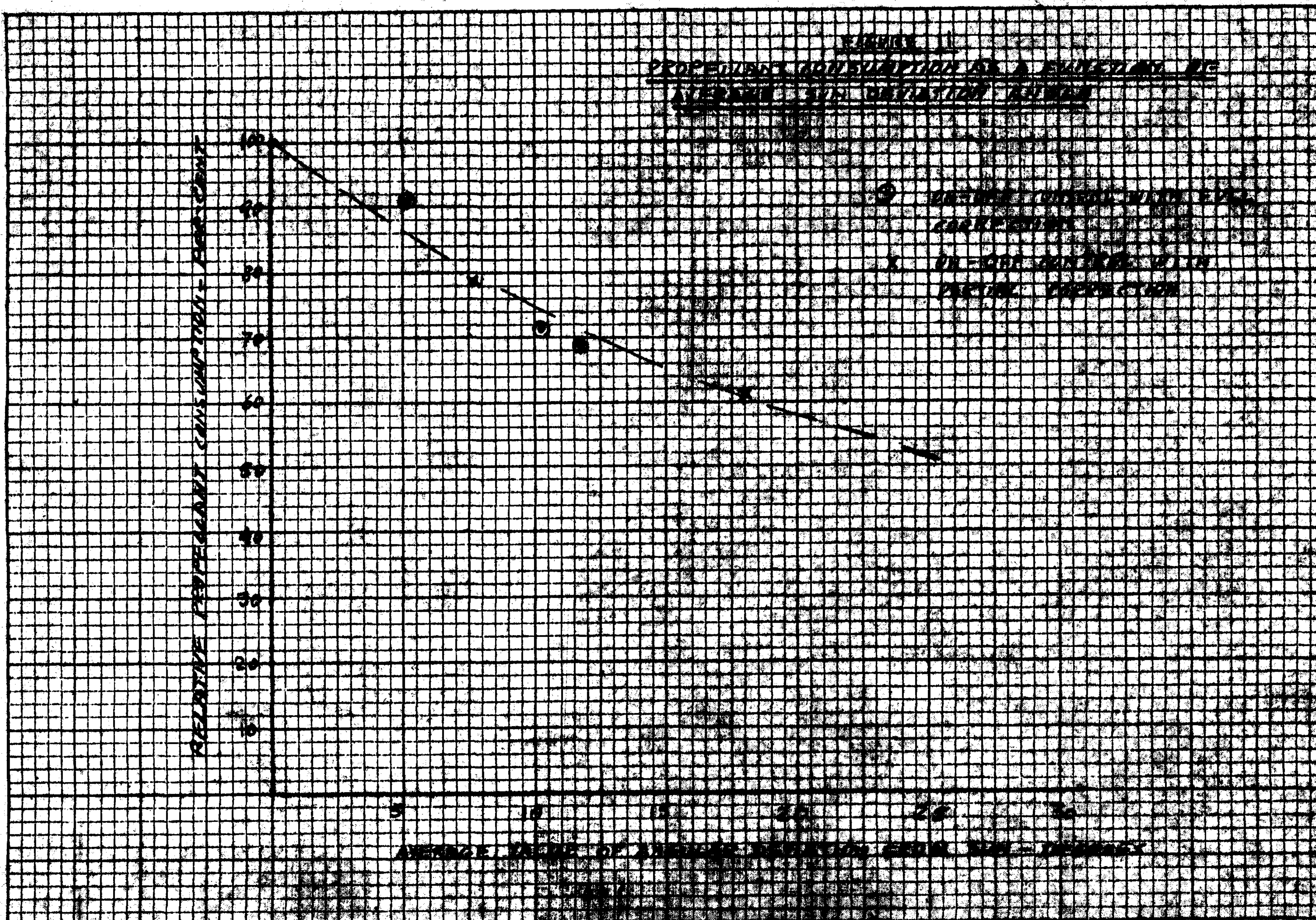
D. Summary

To compare the relative merits of the three control methods discussed above, the various individual results are summarized in Fig. 10. The upper, solid curve indicates the manner in which on-off control with full correction tends to reduce


RELATIVE PROPELLANT CONSUMPTION - PER DAY
0
10
20
30
40
50
60
70
80
90
100
5
10
15
20
25
ANGULAR DEVIATION FROM SUN - DEGREES
RELATIVE PROPELLANT CONSUMPTION
ANGULAR LIMITS SHOWN.
ASSUMPTION THAT SPIN AXIS WILL

propellant consumption as a function of permissible angular misalignment. The lower, dashed curve represents the lower limit for continuous control along an optimum path. The two data points for on-off control with partial correction are also shown, to indicate the effectiveness of this approach relative to the other two. It is immediately evident that the partial correction control policy is most desirable in that it produces a significant reduction in propellant requirements with minimum instrumentation complexity.

In Fig. 11 an attempt is made to generalize upon the limited data evolved in the study of the two on-off control methods. The basis for this generalization is the conjecture that the reduction in yearly propellant requirements is a function of the average deviation angle from the sun. Fig. 11 suggests this to be the case. Should this be true, then the same result would be obtained whether one corrected from, say, 20 degrees to zero, or from 15 degrees to 5 degrees, since the average value (10°) is the same in both cases. Carrying this further, if a maximum sun deviation of, say, 20 degrees is to be tolerated, the greatest benefit would accrue from on-off control between limits as close to 20° as possible. It is to be emphasized, however, that the present study does not offer any proof for the validity of the above relationship and it is more likely that the greatest reduction would be obtained by variable ranges of correction depending upon the time of year in which they are to be applied. This implies variable control gain, but the added complexity is minimal since the maximum frequency of corrections is in the order of once a day.


PROPELLANT CONSUMPTION AS A FUNCTION OF
RELATIVE PROPELLANT CONSUMPTION - PER CENT
100
90
80
70
60
50
40
30
20
10
5
10
15
20
25
30

V CONCLUSIONS

The following conclusions are applicable to a low altitude space station having axial symmetry and a high spin angular momentum about the axis of symmetry.

A. Effect of Launch Parameters

There is a significant variation in the rates of propellant requirement between the two halves and the four quarters of any one year if on-off control is used to orient the station to the sun. This is due to the manner in which the effect of annual motion of the earth around the sun combines with the secular gravity-gradient torque. Neglecting the annual rotation results in an essentially constant average rate of propellant utilization for periods longer than a few weeks.

A time interval of one year is a fundamental period in that it averages out the effects due to earth's motion around the sun, which are related to the variable angle between the equatorial plane and the ecliptic plane. Also one year is long enough to average out the variations occurring during each regression period. Therefore, propellant requirements for a period of one year will be essentially independent of the choice of initial launch conditions. However, for mission durations other than one year, this will not be true.

For the assumed orbit and station parameters, and assuming on-off control to a permissible deviation angle of 20 degrees, propellant requirements for an 18 months mission could, depending upon launch parameters, be as much as 1,475 lbs. or as little as 800 lbs., i.e. a difference of 675 lbs. For a three months mission the effect of launch parameters is even more significant since the maximum requirement could be as high as 900 lbs. whereas the minimum possible is about 210 lbs, viz. a difference of 690 lbs.

The desired launch parameters are defined by the time of the year of orbit injection (t_o) and the time of day associated with a particular launch co-ordinate (γ_o). For the assumed station parameters, the time of day will locate the initial condition within the 53-day cycle of the secular torque. Since this gives a launch window on each day during a period of nearly two months, the desired launch parameters can be selected without greatly restricting the date of launch.

B. Control Criteria for Reaction Jet Actuation

This study has shown that the most practical approach to the control of the station toward the sun is that of using on-off reaction jet actuation to maintain the spin-axis near the maximum permissible angular deviation from the sun in the direction to minimize the secular gravity-gradient torque. It is to be emphasized that although the study is conclusive as regards the superiority of this approach relative to other possible control methods, a nominal amount of further study is required to transform it into a specific control law useable in control system design. The general effectiveness of the recommended approach can be assessed as follows:

Although the results of this study are not necessarily restricted to large space stations, all numerical estimates have been based upon parameters derived from a 21-man station design. We therefore assume that the station will require in the order of 20 kw. of electrical power which is to be supplied by solar cells. To make effective use of the recommended control method we will assume that a maximum of 20 degrees deviation from the sun is permissible. This leads to a 6% maximum loss of electrical power produced by solar cell misalignment. At 100 lbs. per kw., the weight penalty in solar cell panels to assure a minimum of 20 kw. at all times is 130 lbs. From the data of Fig. 11 it is estimated that if the station spin-axis is maintained between 21 and 19 degrees the required propellant for a one-year period will be about 58% of the yearly requirement for continuous sun alignment. The net saving, allowing for the 120 lbs. penalty in extra solar cell panels, is about 1200 lbs., viz. 3,180 lbs. for $\delta=0^{\circ}$ compared to (1,845 + 130) lbs. for $\delta = 20^{\circ}$. If a permissible sun deviation angle of 25 degrees is to be allowed, the net saving relative to continuous sun alignment is 1,380 lbs.

This study indicates that control to within a reasonably narrow band about the maximum permissible misalignment, say to $\pm 1^{\circ}$, is likely to lead to the most effective utilization of the allowable sun deviation. This appears to be compatible with power supply operation since it would minimize fluctuation of power output from the solar cells due to changes in orientation relative to the sun.

The merits of the above control method are the simplicity with which it can be implemented on the one hand and its effectiveness in reducing propellant requirements on the other. Compared to on-off control with full corrections to sun alignment, it differs only in the gain setting which determines the magnitude of the applied control impulse. However, it makes more effective use of the available sun deviation angle as regards propellant consumption.

In its simplest form, implementation of the recommended control policy requires sun sensors and control circuitry which will return the spin axis toward the sun by a fixed amount whenever the limiting angular deviation is reached. This corresponds to the method used in the computer program during the study. In its ultimate form it may be expected that the gain setting, i.e. the setting of control circuitry which determines the amount by which the spin-axis is moved toward the sun, will vary according to a predetermined program. The selection of the desired gain settings as a function of time of year can be made once station parameters and the actual launch time are known. Since changes in gain coefficient are expected to be infrequent, the required computations can be done on the ground and the program of gain coefficients either stored on-board the vehicle if a preselected launch time is met, or transmitted to the station if significant launch delays are encountered. In either case, there appears to be no need for an on-board gravity-gradient computer.

Although a numerical comparison cannot be made with the alternate approach in which the spin axis is made to follow continuously an optimum path within the limits of the allowable deviation angle, the study indicates that such a sophisticated control method is not likely to lead to appreciable propellant saving compared to the recommended method. Since the formulation of the optimum path involves considerable analytical complexity and, more significantly, because its implementation would lead to greatly increased instrumentation complexity, its further study is not warranted.

VI RECOMMENDATIONS

Although this study is conclusive in the choice of a criterion for the attitude control of an axisymmetric, sun-oriented space station, a nominal amount of additional work is needed to render it applicable to control system design. Specifically, it is recommended that the very limited data evolved in this study for on-off control with partial corrections be expanded so as to establish the amount of correction which is most effective at a given time of the year. It appears likely that variable gain settings might further reduce propellant requirements without adding to the complexity of on-board instrumentation; since corrections are relatively infrequent and since the desired gain values can be established in advance, gain changes can be made manually.

From experience gained in the course of this study it appears possible to simplify the various analytical relationships for the purpose of the further study recommended above. Thus, the simplified relationship will, at best, yield an analytical derivation of the detailed control laws which are sought and in any case they will serve as useful guidelines for computer studies aimed at the formulation of the control laws.

In view of the potential utility of the control criterion evolved in this study, it is also recommended that further work be done to extend it to stations of arbitrary mass distribution.

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5. Computer data provided by Mr. Roy G. Woodle of Minneapolis-Honeywell from the study of reference 4

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APPENDIX A: DERIVATION OF BASIC EQUATIONS

1. The Gravity-Gradient Torque

A concise derivation of the gravity-gradient torque using the nomenclature of Figure 1 is more pertinent than a transformation of results given in the literature (Ref. 1, 2, 3). The procedure is to calculate the force on a differential particle of mass assuming a spherical earth, evaluate the differential torque about the station center of mass, and then sum over all particles in the system.

Employing the nomenclature of Fig. 1, the differential force and torque are:

$$d^3\vec{F} = -\frac{GM}{R^3}\, dm\, \vec{R} \quad , \quad dm = \rho \cdot d\xi \cdot d\eta \cdot d\zeta \tag{1}$$

$$d^3\vec{L} = \vec{r} \times d^3\vec{F} = -\frac{GM}{R^3}\, dm\, \vec{r} \times \vec{R} \tag{2}$$

where G is the gravitational constant and M is the earth mass.

Since:

$$\vec{R} = \vec{R}_c + \vec{r} \quad , \quad \vec{r} \times \vec{r} = 0 \tag{3}$$

We have:

$$d^3\vec{L} = -\frac{GM}{R^3}\, dm\, R_c\, \vec{r} \times e \quad ; \quad \vec{e} = \frac{\vec{R}_c}{R_c} \tag{4}$$

where $\vec{e}$ is a unit vector along the station or local vertical.

Using the usual expansion for R^{-3} namely:

$$R^2 = \vec{R} \cdot \vec{R} = R_c^2 \left(1 + 2\,\frac{\vec{r} \cdot \vec{R}_c}{R_c^2}\right) \quad ; \quad \frac{r}{R_c} << 1$$

$$R^{-3} = \frac{1}{R_c^3}\left(1 - 3\,\frac{\vec{r} \cdot \vec{R}_c}{R_c^2}\right) \tag{5}$$

the torque (4) may be written:

$$d^3\vec{L} = -\frac{GM}{R_c^2}\, dm \left(1 - 3\,\frac{\vec{r} \cdot \vec{e}}{R_c}\right)\left(\vec{r} \times \vec{e}\right) \tag{6}$$

Integrating (6) over the station, the first term gives zero since $\vec{r}$ has its origin at the station center of mass and $\vec{e}$ is constant; hence we are left with:

$$\vec{L} = -3\omega_0^2\, \vec{e} \times \int_{\text{BODY}} \vec{r}\; \vec{r}\cdot\vec{e}\; \rho\, dv \tag{7}$$

$$\omega_0^2 = \frac{GM}{R_0^3} \quad = \text{station circular orbital angular speed squared} \tag{8}$$

The $\vec{r}\,\vec{r}$ form of (7) may be exploited to yield $\vec{L}$ directly in terms of the inertia dyad plus other terms, but we shall maintain the simpler vector approach and use the relationships:

$$\vec{e} = \vec{\xi}\; \vec{e}\cdot\vec{\xi} + \vec{\eta}\; \vec{e}\cdot\vec{\eta} + \vec{\zeta}\; \vec{e}\cdot\vec{\zeta} \tag{9}$$

$$\vec{r} = \vec{\xi}\,\xi + \vec{\eta}\,\eta + \vec{\zeta}\,\zeta \tag{10}$$

$$\vec{e}\cdot\vec{r} = \xi\; \vec{e}\cdot\vec{\xi} + \eta\; \vec{e}\cdot\vec{\eta} + \zeta\; \vec{e}\cdot\vec{\zeta} \tag{11}$$

$$\begin{aligned}\vec{e}\times\vec{r} = \vec{\xi}\,(\zeta\, \vec{e}\cdot\vec{\eta} - \eta\, \vec{e}\cdot\vec{\zeta}) + \vec{\eta}\,(\xi\, \vec{e}\cdot\vec{\zeta} - \zeta\, \vec{e}\cdot\vec{\xi}) \\ + \vec{\zeta}\,(\eta\, \vec{e}\cdot\vec{\xi} - \xi\, \vec{e}\cdot\vec{\eta})\end{aligned} \tag{12}$$

where $\vec{\xi}, \vec{\eta}, \vec{\zeta}$ are to be considered as unit vectors.

The integrand of (7) becomes accordingly:

$$\begin{aligned}(\vec{e}\times\vec{r})(\vec{e}\cdot\vec{r}) = \vec{\xi}\,\Big\{&(\vec{e}\cdot\vec{\zeta})(\vec{e}\cdot\vec{\eta})(\zeta^2+\xi^2-\eta^2-\xi^2) \\ &-\xi\eta\,(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\zeta}) + \xi\zeta\,(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\eta}) \\ &+\eta\zeta\,[(\vec{e}\cdot\vec{\eta})^2 - (\vec{e}\cdot\vec{\zeta})^2]\Big\} \\ + \vec{\eta}\,\Big\{&(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\zeta})(\xi^2+\eta^2-\zeta^2-\eta^2) + \xi\eta\,(\vec{e}\cdot\vec{\eta})(\vec{e}\cdot\vec{\zeta}) \\ &-\eta\zeta\,(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\eta}) + \xi\zeta\,[(\vec{e}\cdot\vec{\zeta})^2 - (\vec{e}\cdot\vec{\xi})^2]\Big\} \\ + \vec{\zeta}\,\Big\{&(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\eta})(\eta^2+\zeta^2-\xi^2-\zeta^2) + \eta\zeta\,(\vec{e}\cdot\vec{\xi})(\vec{e}\cdot\vec{\zeta}) \\ &+\xi\zeta\,(\vec{e}\cdot\vec{\eta})(\vec{e}\cdot\vec{\zeta}) + \xi\eta\,[(\vec{e}\cdot\vec{\xi})^2 - (\vec{e}\cdot\vec{\eta})^2]\Big\}\end{aligned} \tag{13}$$

Upon performing the integration in (7) the various inertias and products of inertia emerge i.e.:

$$I_{\xi\xi} = \int \rho\,(\eta^2+\zeta^2)\,dv$$

$$I_{\xi\eta} = \int \rho\,\xi\eta\,dv \qquad \text{etc.}$$

Now if $\xi\eta\zeta$ are a set of principal axes then all products of inertia terms vanish, and we have:

$$\vec{L} = -3\omega_0^2 \Big\{ \vec{e}\cdot\vec{\xi} \left[\vec{\zeta}(\vec{e}\cdot\vec{\eta})(I_{\eta\eta} - I_{\xi\xi}) + \vec{\eta}(\vec{e}\cdot\vec{\zeta})(I_{\xi\xi} - I_{\zeta\zeta}) \right] + \vec{\xi}(\vec{e}\cdot\vec{\zeta})(\vec{e}\cdot\vec{\eta})(I_{\zeta\zeta} - I_{\eta\eta}) \Big\} \tag{14}$$

If, in addition, there is axial symmetry about $I_{\xi\xi}$ (to be identified as the spin axis) such that:

$$I_{\zeta\zeta} = I_{\eta\eta} = I \ ; \tag{15}$$

$$I_{\xi\xi} = I_s \ , \quad I_s - I = \Delta I \tag{16}$$

then (14) reduced to:

$$\vec{L} = 3\omega_0^2 \Delta I \, (\vec{e}\cdot\vec{\xi}) \left[\vec{\zeta}(\vec{e}\cdot\vec{\eta}) - \vec{\eta}(\vec{e}\cdot\vec{\zeta}) \right] \tag{17}$$

From (9) we note that:

$$\vec{e} \times \vec{\xi} = \vec{\zeta}\,\vec{e}\cdot\vec{\eta} - \vec{\eta}\,\vec{e}\cdot\vec{\zeta}$$

Hence, we have finally:

$$\vec{L} = 3\omega_0^2 \Delta I \, (\vec{e}\cdot\vec{\xi})(\vec{e}\times\vec{\xi}) \equiv C_0 (\vec{e}\cdot\vec{\xi})(\vec{e}\times\vec{\xi}) \tag{18}$$

$$|\vec{L}| = \tfrac{1}{2} C_0 \,\mathrm{SIN}\, 2 \,(\text{angle between } \vec{e} \text{ and } \vec{\xi}) \tag{19}$$

Before continuing to the secular torque or average torque per orbit we note from the general expression (7) that:

$$\vec{e}\cdot\vec{L} = \vec{e}\cdot\left[-3\omega_0^2\, \vec{e}\times\vec{A}\right] \equiv 0 \quad ; \quad \vec{A} \equiv \int_{\text{BODY}} \vec{r}\ \vec{r}\cdot\vec{e}\ \rho\, dv$$

Hence, there is the general rule that the instantaneous gravity-gradient torque is always perpendicular to the line between the center of the earth and the vehicle center of mass.

If the spin velocity ω_s about I_S is sufficiently high the station will not respond to the variations in torque about the orbital path, and it is only necessary to consider the average torque per orbit. Assume then that the spin axis $\vec{\xi}$ does not change over an orbit. Define a new right-handed orthogonal coordinate system $\vec{u}, \vec{v}, \vec{w}$ (Fig. 1) such that $\vec{u}$ is perpendicular to the orbital plane, and $\vec{w}$ lies along the intersection of the orbital plane and the plane passing through $\vec{\xi}$ and perpendicular to the orbital plane. In this system we have:

$$\vec{e} = \vec{v}'\,\vec{e}\cdot\vec{v}' + \vec{w}'\,\vec{e}\cdot\vec{w}' = \vec{v}'\cos(\omega_o t + \theta_1) + \vec{w}'\sin(\omega_o t + \theta_1) \tag{20}$$

where θ_1 is an arbitrary phase angle. Substituting (20) into (18) noting that $\vec{v}'\cdot\vec{\xi}$ is zero since $\vec{\xi}$ lies in the $\vec{u}, \vec{w}'$ plane, and averaging over an orbit gives:

$$\vec{L}_s = \frac{2\pi}{\omega_o} C_o (\vec{w}'\cdot\vec{\xi})(\vec{w}'\times\vec{\xi}) \int_0^{2\pi/\omega_o} \sin^2(\omega_o t + \theta_1)\,dt$$

$$= \tfrac{1}{2} C_o (\vec{w}'\cdot\vec{\xi})(\vec{w}'\times\vec{\xi}) = \vec{v}'\,\frac{C_o}{4}\sin 2a' \tag{21}$$

$$\text{or} \quad \vec{L}_s = \tfrac{1}{2} C_o (\vec{\xi}\cdot\vec{u})(\vec{\xi}\times\vec{u}) \tag{22}$$

where it is to be remembered that $\vec{w}', \vec{\xi}, \vec{u}$ are all unit vectors, and $\vec{u}$ is given by: ($\vec{i}, \vec{j}, \vec{k}$ are unit vectors in the xyz system)

$$\vec{u} = -\vec{i}\sin i\cos\psi + \vec{j}\cos i + \vec{k}\sin i\cdot\sin\psi$$

$$= -\vec{i}\cos\beta\cos a + \vec{j}\sin\beta\cos a + \vec{k}\sin a \tag{23}$$

A special case of interest is that for which $\vec{\xi}$ points in the direction of the sun (z-axis, $\vec{k}$ unit vector). The $\vec{u}, \vec{v}', \vec{w}'$ system becomes the $\vec{u}, \vec{v}, \vec{w}$ system with equations:

$$\vec{L}_s^s = \frac{C_o}{2}(\vec{w}\cdot\vec{k})(\vec{w}\times\vec{k}) = \frac{C_o}{2}(\vec{k}\cdot\vec{u})(\vec{k}\times\vec{u}) = \vec{v}\,\frac{C_o}{4}\sin 2a \tag{24}$$

$$\vec{v} = -\vec{i}\sin\beta - \vec{j}\cos\beta = -(1+\tan^2 i\cdot\cos^2\psi)^{-1/2}(\vec{i} + \vec{j}\tan i\cdot\cos\psi) \tag{25}$$

$$= -\frac{1}{\cos a}(\vec{i}\cos i + \vec{j}\sin i\cdot\cos\psi)$$

$$\vec{w} = \vec{i}\cos\beta\sin a - \vec{j}\sin\beta\sin a + \vec{k}\cos a$$

$$= \vec{i}\tan a\cdot\sin i\cdot\cos\psi - \vec{j}\tan a\cdot\cos i + \vec{k}\cos a \tag{26}$$

The relationship among the angles are as follows:

$$\sin a = \sin i\cdot\sin\psi \;; \qquad \cos a = \frac{\cos i}{\sin\beta} \tag{27}$$

$$\sin\beta = (1+\tan^2 i\cdot\cos^2\psi)^{-1/2} \;; \qquad \cos\beta = \frac{\sin i\cdot\cos\psi}{\cos a}$$

II. The Equations of Motion for the Spinning Symmetrical Station

If the station is "stiff" so that $\vec{\xi}$ does not change appreciable over an orbit, and the angular momentum about the spin axis $\vec{\xi}$ is large compared with angular momentum arising from gravity-gradient torque, then:

$$d\vec{H}_s/dt = \vec{L}_s$$

$$I_s\,\omega_s \frac{d\vec{\xi}}{dt} = \frac{C_0}{2}(\vec{\xi}\cdot\vec{u})(\vec{\xi}\times\vec{u}) \qquad (28)$$

Choosing an XYZ inertial system with X along the autumnal equinox and Y perpendicular to the ecliptic, we have:

$$\vec{\xi} = \vec{I}\,\xi_x + \vec{J}\,\xi_Y + \vec{K}\,\xi_Z \qquad (29)$$

$$\vec{i} = \vec{I}\cos\psi_s - \vec{K}\sin\psi_s \;; \qquad \vec{j} = \vec{J} \qquad (30)$$
$$\vec{k} = \vec{I}\sin\psi_s + \vec{K}\cos\psi_s$$

Substituting (30) into (23) we have for $\vec{u}$:

$$\vec{u} = -\vec{I}\sin i\cos(\psi+\psi_s) + \vec{J}\cos i + \vec{K}\sin i\cdot\sin(\psi+\psi_s) \qquad (31)$$

$$\vec{u} \equiv \vec{I}\,u_x + \vec{J}\,u_Y + \vec{K}\,u_Z \qquad (32)$$

From (28) and (32) the equations of motion become:

$$\frac{d\xi_x}{dt} = K_1\, f(t)\, g_1(t)$$
$$\frac{d\xi_Y}{dt} = K_1\, f(t)\, g_2(t) \quad ; \quad \frac{d\xi_Z}{dt} = K_1\, f(t)\, g_3(t) \qquad (33)$$

where:

$$K_1 \equiv \frac{C_0}{2 I_s \omega_s} \qquad (34)$$

$$f(t) \equiv \vec{\xi}\cdot\vec{u} = \xi_x u_x + \xi_Y u_Y + \xi_Z u_Z \qquad (35)$$

$$g_1(t) \equiv (\vec{\xi}\times\vec{u})_x = \xi_Y u_Z - \xi_Z u_Y \qquad (36)$$

$$g_2(t) \equiv (\vec{\xi}\times\vec{u})_Y = \xi_Z u_x - \xi_x u_Z \qquad (37)$$

$$g_3(t) = (\vec{\xi} \times \vec{u})_z = \xi_x u_Y - \xi_Y u_x \tag{38}$$

$$u_x \equiv -\sin i \cdot \cos(\psi + \psi_s)\,; \quad u_Y \equiv \cos i\,; \quad u_z \equiv \sin i \cdot \sin(\psi + \psi_s) \tag{39}$$

The angular misalignment, δ, between the spin axis and the sun line is given by:

$$\delta = \cos^{-1}(\vec{\xi} \cdot \vec{k}) = \cos^{-1}(\xi_x \sin\psi_s + \xi_z \cos\psi_s) \tag{40}$$

APPENDIX B: POSITION OF SPIN AXIS AT A GIVEN ANGULAR MISALIGNMENT FOR WHICH THE SECULAR GRAVITY GRADIENT TORQUE IS MINIMUM

Consider the spin axis $\vec{\xi}$ at some angular misalignment δ with respect to the sun line Z in Fig. 1. It will be shown that the magnitude of the secular gravity-gradient torque is a minimum if the spin axis lies in the $\vec{\xi}, \vec{\omega}$ plane rather than along any other ray of the cone with semi-vertex angle δ.

From equation (21) of Appendix A the secular torque magnitude is:

$$|L_s| = \frac{C_0}{4} \sin 2a' = \frac{C_0}{2} \sin a' \cdot \cos a' = \frac{C_0}{2} \sin a' \sqrt{1 - \sin^2 a'} \tag{1}$$

From (23) of Appendix A:

$$\sin a' = \vec{\xi} \cdot \vec{u} = -\xi_x \sin i \cdot \cos \varphi + \xi_y \cos i + \xi_z \sin a \tag{2}$$

Since:

$$\xi_x^2 + \xi_y^2 + \xi_z^2 = 1 \quad ; \quad \xi_z^2 = \cos^2 \delta$$

Then:

$$\xi_y^2 = \sin^2 \delta - \xi_x^2 \tag{3}$$

Hence $\sin a'$ may be written as:

$$\sin a' = -\xi_x \sin i \cos \varphi + \sqrt{\sin^2 \delta - \xi_x^2} \cos i + \cos \delta \cdot \sin a \tag{4}$$

To find the minimum condition for $|\vec{L}_s|$ we have from (1):

$$\frac{d|\vec{L}_s|}{d\xi_x} = 0 = \sin a' \frac{1}{2} \frac{1}{\sqrt{1 - \sin^2 a'}} (-2 \sin a') \frac{d \sin a'}{d \xi_x} + \sqrt{1 - \sin^2 a'} \frac{d \sin a'}{d \xi_x} \tag{5}$$

From (5) we have two conditions:

$$\sin^2 a' = \cos^2 a' \quad ; \quad a' = 45^\circ$$

or:

$$\frac{d \sin a'}{d \xi_x} = 0 \tag{6}$$

Substituting (4) into (6) we get:

$$-\sin i \cdot \cos\psi + \cos i \, \frac{1}{2} \frac{-2\xi_x}{\sqrt{\sin^2\delta - \xi_x^2}} = 0$$

$$\frac{\xi_x}{\xi_y} = -\tan i \cdot \cos\psi = -\cot\beta \tag{7}$$

From (3) and (7) we have:

$$\xi_y^2 = \frac{\sin^2\delta}{1+\cot^2\beta} = \sin^2\delta \sin^2\beta$$

$$\xi_y = \mp \sin\delta \sin\beta \tag{8}$$

$$\xi_x = \pm \sin\delta \cos\beta \tag{9}$$

where the top signs place $\vec{\xi}$ below the ecliptic and the bottom ones correspond to above the ecliptic. Using equations (27) of Appendix A and equations (8) and (9) in (2) we have:

$$\sin a' = \mp \sin\delta \cos a \left(\cos^2\beta + \sin^2\beta\right) + \cos\delta \sin a$$

$$\sin a' = \sin(a \mp \delta) \tag{10}$$

Equation (7) establishes that $\vec{\xi}$ is in the $\vec{z}, \vec{w}$ plane while (10) in conjunction with (1) gives the torque magnitude:

$$|M_z| = \frac{C_0}{2} \sin 2(a \mp \delta) \tag{11}$$

The $\pm$ signs correspond to $\vec{\xi}$ and the orbital plane below and above the ecliptic at z respectively. That (11) is a minimum torque and not maximum is evident from the geometry.